Probability Density Function Expressed in Terms of Hankel Transform and Its Application to Random Flight Problem

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(Received Apr. 16, 1968)

In this paper a formula of probability density function expressed in terms of Hankel transform is introduced. The application of the formula to a problem of generalized multidimensional random flights and some other physical phenomena where random fluctuation is of main concern is discussed in some details.

I. Introduction

In his "Cybernetics", N. Wiener[1] emphasizes his standpoint that almost all phenomona contain in an explicit sense random and irreversible processes which need statistical treatment, and he enumerates facts illustrating his theory in various fields of "gravitational astronomy", "meteorology", "power engineering", "communication engineering", "biology", etc. Mathematical models of the random process such as "ruin problem", "diffusion process", "random walks", "asymptotic distribution", "Wiener process", "Brownian motion", etc. have been investigated by Pearson, Rayleigh, Kluyver, Treloar, Chandrasekhar, Uhlenbeck, Langevin, Ornstein, Smoluchowski, Rice, Nakagami, weyl, etc., as mentioned in the previous paper.

Let us now try to illustrate the irreversible process. When a random process undergoes a kind of "averaging operation" one can see an irreversible property come into the picture. Suppose a random process has a certain mean value. No one can assert this process is a unique one which has the particular mean value, since there can possibly be many distinct random processes having identically the same mean value.

In this paper a formula of probability density function expressed in terms of Hankel transform of order ν is introduced and a general expression and its application to the problem of multidimensional random flights and some other physical phenomena are given. Because of limited space details of calculation will be omitted to avoid lengthiness.

II. An n-dimensional probability density function in a form of Hankel transform of order ν

Suppose variables, x_k $(K=1,2,\cdots\cdots,N)$ are governed by a certain probability density function $P(x_1,\cdots\cdots,x_N)$ assigned a priori. One may ask: What is the probabiblity density function of R or E $(=R^2)$ when

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$$R^2 = \sum_{k=1}^{N} x_k^2 \text{ (i. e., hypersphere) ?} \qquad \cdots\cdots\cdots(1)$$

Expanding a random fluctuation f (t) in an orthonormal series, one can write

$$\left.\begin{aligned} f(t) &= \sum_i f_i \cdot \varphi_i(t), \\ f_i &= \int_{t_1}^{t_2} f(t)\varphi_i(t)^*\rho(t)dt \equiv (f, \varphi_i), \end{aligned}\right\} \qquad \cdots\cdots\cdots(2)$$

and from Parseval equation the norm $||f||$ becomes

$$||f|| = \sum |f_i|^2 \ (||f||^2 \equiv (f, f)). \qquad \cdots\cdots\cdots(3)$$

Now the problem of determining the probability density P(f) from P (f_i) given a priori is substantially equal to the previous question. Let $\{\varphi_i(t)\}$ be trigonometric series or $\left\{\dfrac{\sin(2\pi wt - \pi n)}{2\pi wt - \pi n.}\right\}$. Using Fourier analysis and Fourier single integral theorem, one obtains, respectively from Eqs. (2) and (3), the following expressions, one due to S. O. Rice[4)8)] and the other due to C. E. Shannon. [3)8)]

$$\left.\begin{aligned} f(t) &= \frac{a_0}{2} + \sum_n \left(a_n \cos\frac{2\pi n}{T}t + b_n \sin\frac{2\pi n}{T}t\right) \\ \frac{1}{T}\int_0^T f(t)^2 dt &= \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}(a^2_n + b^2_n) \end{aligned}\right\} \qquad \cdots\cdots\cdots(4)$$

$$\left.\begin{aligned} f(t) &= \sum_n x_n \frac{\sin(2\pi wt - \pi n)}{2\pi wt - \pi n} \\ \frac{1}{T}\int_{-T/2}^{T/2} f(t)^2 dt &= \frac{1}{2TW}\sum_{n=1}^{2TW} x_n^2 \ ; \ x_n \equiv f\left(\frac{n}{2W}\right) \end{aligned}\right\} \qquad \cdots\cdots\cdots(5)$$

Here in Eq. (5) f(t) is taken to be zero for time and frequency outside the intervals T and W.

In general a joint probability distribution Q ($R_1, \cdots\cdots, R_k$) can be expressed as[7)10)]

$$\left.\begin{aligned} Q(R_1, \cdots\cdots, R_k) &= \int_0^\infty \cdots \int_0^\infty P(r_1, \cdots\cdots, r_k) \prod_{j=1}^{K} D_j(r_j) dr_j, \\ D_j(r_j) &= 1 (r_j < R_j) \ ; \ = 0 (r_j > R_j). \end{aligned}\right\} \qquad \cdots\cdots\cdots(6)$$

Using the discontinuous integral by Weber-Schafheitlin[5)] :

$$R^m \int_0^\infty J_m(R\lambda) \frac{J_{m-1}(r\lambda)}{r^{m-1}} d\lambda = \left.\begin{matrix} 1 \ (r<R) \\ 0 \ (r>R) \end{matrix}\right\} \qquad \cdots\cdots\cdots(7)$$

in place of D_j (r_j), one obtains for K = 1 and m $\geqslant$ 1/2

$$P(R) = \frac{dQ(R)}{dR} = R^m \cdot \int_0^\infty \lambda J_{m-1}(\lambda R) \overline{\frac{J_{m-1}(\lambda r)}{r^{m-1}}} d\lambda, \qquad \cdots\cdots\cdots(8)$$

or more explicitly

$$P(R) = \frac{R^m}{2^{m-1}.\Gamma(m)} \int_0^\infty F(\lambda)\lambda^m J_{m-1}(\lambda R) d\lambda, \qquad \cdots\cdots\cdots(9)$$

$$\left.\begin{aligned} F(\lambda) &= \frac{2^{m-1}.\Gamma(m)}{\lambda^{m-1}} \int_0^\infty \frac{J_{m-1}(\lambda R)}{R^{m-1}} P(R) dR \\ &\equiv \Gamma(m) \overline{\frac{J_{m-1}(\lambda R)}{(\lambda R/2)^{m-1}}} = 1 + \sum_n \frac{(-1)^n \Gamma(m) \Omega_n}{2^{2n} n! \Gamma(m+n)} \lambda^{2n}, \\ \lim_{\lambda \to 0} F(\lambda) &= 1. \end{aligned}\right\} \qquad \cdots\cdots\cdots(10)$$

These relations show that P(R) is uniquely determined by $\Omega_n \equiv \overline{R^{2n}}\ (=\overline{E^n})$. By applying N-dimensional polar coordinate transformations[10] :

$$(x_1,\cdots\cdots,x_N)\to(R,\theta_1,\cdots\cdots,\theta_{N-1}),\quad (\mu_1,\cdots\cdots,\mu_N)\to(\lambda,\varphi_1,\cdots\cdots,\varphi_{N-1})$$

to $P(x_1,x_2\cdots\cdots,x_N)$ in Eq. (1) and to its characteristic function $F(\mu_1,\cdots\cdots,\mu_N)$, one can also derive Eq. (9) for $m = N/2$ (use Eq. (32)) and

$$\left.\begin{aligned} &F(\lambda) = \frac{1}{S_{(N)}}\int_0^{2\pi}\int_0^{\pi}\cdots\cdots\int_0^{\pi}[F(\mu_1,\cdots\cdots,\mu_N)]\ d\&_{(N)}, \\ &\qquad\qquad (\mu_1,\cdots\cdots,\mu_N)\to(\lambda,\varphi_1,\cdots\cdots,\varphi_{N-1}) \\ &F(\mu_1,\cdots\cdots,\mu_N) \equiv \overline{\exp\{i\textstyle\sum x_k\mu_k\}}, \\ &\text{where} \\ &S_{(N)} \equiv \frac{\pi^{N/2}\cdot N}{\Gamma\left(\frac{N}{2}+1\right)},\quad d\&_{(N)} \equiv \prod_{j=1}^{N-1}(\sin\varphi_j)^{N-j-1}\ d\varphi_j. \end{aligned}\right\} \quad \cdots\cdots\cdots(11)$$

$F(\lambda)$ in Eqs. (11) is found to be equivalent to Eq. (10) from the inversion formula of Hankel transform of order $\nu\ (=m-1)$. $S_{(N)}$ and $d\&_{(N)}$ denote respectively the surface area and incremental surface area of unit hypersphere.

III. General expressions for random phenomena: Bessel distribution

If x_k's in Eq. (1) are normally distributed with the joint density function:

$$P(x_1,\cdots\cdots,x_N) = \prod_{i=1}^{N}\frac{1}{\sqrt{2\pi}\ \sigma}e^{-\frac{(x_i-A_i)^2}{2\sigma^2}}$$

$$\equiv\prod_{i=1}^{N} n(x_i;A_i,\sigma^2);A_i=\delta_{i1}\cdot A\ , \quad \cdots\cdots\cdots(12)$$

where A is a constant (or stationary part) which corresponds to $a_0/2$ in Eq. (4), the amplitude characteristic function $F(\lambda)$ in Eqs. (11) can be written explicitly as

$$F(\lambda) = e^{-\frac{\Omega_0}{4m}\lambda^2}\cdot\frac{\Gamma(m)\ 2^{m-1}}{(\lambda A)^{m-1}}J_{m-1}(\lambda A),$$

$$(A^2\equiv E_0=\Omega_1-\Omega_0),\quad m=N/2. \quad \cdots\cdots\cdots(13)$$

Substituting Eq. (13) into Eq. (9), one obtains[2][7]

$$P(R) = \frac{2mR^m}{\Omega_1-A^2}\left(\frac{1}{A}\right)^{m-1}\cdot e^{-\frac{m(R^2+A^2)}{\Omega_1-A^2}}\ I_{m-1}\left(\frac{2mAR}{\Omega_1-A^2}\right), \quad \cdots\cdots\cdots(14)$$

$$Q(R) = e^{-\frac{m(R^2+A^2)}{\Omega_1-A^2}}\cdot\sum_{n=0}^{\infty}\left(\frac{R}{A}\right)^{m+n} I_{m+n}\left(\frac{2mAR}{\Omega_1-A_2}\right), \quad \cdots\cdots\cdots(15)$$

$$\overline{R^K} = \left(\frac{\Omega_1-A_2}{m}\right)^{K/2}\cdot\frac{\Gamma(m+K/2)}{\Gamma(m)}{}_1F_1\left(-\frac{K}{2}\ ;\ m\ ;\ -\frac{mA^2}{\Omega_1-A^2}\right), \quad \cdots\cdots\cdots(16)$$

by using an identity:

$$Z^{\nu-K}\cdot I_{\nu-K}(Z) = \left(\frac{d}{ZdZ}\right)^K\{Z^\nu I_\nu(Z)\}$$

and the Kummer's first transformation[5] for the confluent hypergeometric series. For a nonnormal random phenomenon in which Eq. (12) does not hold one may still use Eq. (14) by assuming a slowly varying part of fluctuation to be equal to A. A special case when $Q(A) = (Q(R))_{R=A}$ represents the effect of mixing randomly varying parts on the stationary part A.

$$Q(A)=\frac{1}{2}\left[1-e^{-\frac{2mA^2}{\Omega_1-A^2}}\cdot\sum_{\nu=-(m-1)}^{m-1} I_\nu\left(\frac{2mA^2}{\Omega_1-A^2}\right)\right] \qquad \cdots\cdots\cdots(17)$$

From Eqs. (13) and (14) one can also obtain the following properties[2)7)] :

$$\left.\begin{aligned} P(R) &= \frac{2m^m}{\Gamma(m)\Omega_1^m}R^{2m-1}\cdot e^{-\frac{m}{\Omega_1}R^2} \\ F(\lambda) &= \exp\left(-\frac{\Omega_1}{4m}\lambda^2\right) \\ Q(R) &= \frac{1}{\Gamma(m)}\gamma\left(m,\frac{m}{\Omega_1}R^2\right) \end{aligned}\right\} \text{ for } A\to 0, \qquad \cdots\cdots\cdots(18)$$

$$P(R)\longrightarrow n(R;A,\sigma'^2)(\sigma'^2\equiv\Omega_0/2m) \quad \text{for } A\to\infty. \qquad \cdots\cdots\cdots(19)$$

Corresponding to Eqs. (14) and (18), the probability density P (E) becomes

$$P(E) = \frac{m}{m_T-E_0}\left(\sqrt{\frac{E}{E_0}}\right)^{m-1}\cdot e^{-\frac{m(E+E_0)}{m_T-E_0}} I_{m-1}\left(\frac{2m\sqrt{EE_0}}{m_T-E_0}\right)$$

$$(m_T\equiv\Omega_1,\sigma_T^2\equiv\langle(E-\bar{E})^2\rangle\equiv\langle(E-m_T)^2\rangle)\ , \qquad \cdots\cdots\cdots(20)$$

$$P(E) = \frac{1}{\Gamma(m)}\left(\frac{m_T}{\sigma_T^2}\right)^m\cdot E^{m-1}e^{-\frac{m_T}{\sigma_T^2}E} \xrightarrow[m\to\infty]{} n(E;m_T,\sigma_T^2). \qquad \cdots\cdots\cdots(21)$$

P(R) in Eqs. (18) under the condition (29) (shown below) coincides with an approximate expression derived by Nakagami for two-dimensional case, and P (E) given by Eq. (21) agrees with one suggested by Rice[4)] without proof.
Moreover, Eqs. (14) and (16) give Rice's expressions[4)8)] when $m=1$. Using the normalized variable defined by

$$\left.\begin{aligned} & X=\frac{Sm}{\overline{U^n}-A_u}U^n\ ;\ B=\frac{Sm}{\overline{U^n}-A_U}A_U\ , \\ & A_U\ :\ \text{A Stationary Part of } \overline{U^n} \\ & U=R,\ A_U=A^2 \text{ for } n=2\ ;\ U=E,\ A_U=E_0 \text{ for } n=1, \end{aligned}\right\} \qquad \cdots\cdots\cdots(22)$$

both Eqs. (14) and (20) become

$$P(X) = \frac{1}{S}\left(\sqrt{\frac{X}{B}}\right)^{m-1}\cdot e^{-\frac{1}{S}(X+B)} I_{m-1}\left(\frac{2}{S}\sqrt{XB}\right): \underline{\text{Bessel Distribution}}^{2)7)} \qquad \cdots\cdots\cdots(23)$$

$$=\frac{X^{m-1}}{\Gamma(m)S^m}\cdot e^{-\frac{X}{S}}\cdot\left\{1+\sum_{n=1}^{\infty}\left(-\frac{B}{S}\right)^n\frac{\Gamma(m)}{\Gamma(m+n)}L_n^{(m-1)}\left(\frac{X}{S}\right)\right\}, \qquad \cdots\cdots\cdots(24)$$

$$P(X)=Ke^{-\alpha X}\cdot X^{\nu/2}\ I_\nu(q\sqrt{X}) : \underline{\text{Standard Bessel Distribution}}^{7)}$$

$$P(U) = \frac{nmU^{\frac{n}{2}(m+1)-1}}{\overline{U^n}-A_U}(\sqrt{A_U})^{1-m}\cdot e^{-\frac{m(U^n+A_U)}{\overline{U^n}-A_U}} I_{m-1}\left(\frac{2m\sqrt{U^nA_U}}{\overline{U^n}-A_U}\right), \qquad \cdots\cdots\cdots(25)$$

where $s=$ any positive number, $q=2\sqrt{B}/s$, $\alpha=1/s$, $\nu=m-1$ and $K=(1/\sqrt{B})^{m-1}(1/s)\exp(-B/s)$. The equation (23) has the following properties[2)7)] :

$$m(t)\equiv\int_{-\infty}^{\infty}e^{tX}\,p(X)dX\ \ (p(X)=0,\ X<0)$$

$$=(1-st)^{-m}\cdot\exp\{Bt/(1-st)\}\ (t<1/s), \qquad \cdots\cdots\cdots(27)$$

$$\left.\begin{aligned}&\overline{X} = m\,s + B,\\&\sigma_x^2(:\ \text{variance of } x) = m\,s^2 + 2Bs.\end{aligned}\right\} \quad \cdots\cdots\cdots (28)$$

Similarly Eqs. (18) and (21) become

$$P(x) = \frac{1}{\Gamma(m)S^m} X^{m-1} \cdot e^{-\frac{X}{S}} \left(:\ \Gamma\text{-Distribution};\ X = \frac{smU^n}{\overline{U^n}}\right)$$

$$\xrightarrow[m\to\infty]{} n(x : \overline{x}, \sigma_x^2),$$

$$\left[P(U) = \frac{nm^m}{\Gamma(m)\ (\overline{U^n})^m} U^{nm-1} \cdot e^{-\frac{mU^n}{\overline{U^n}}}\right], \quad \cdots\cdots\cdots (26)$$

which is the first term of Eq. (24) ($A=0$), a Laguerre series.[6] The last expression for $P(x)$ (or $P(R)$) gives a good approximation for up to the second moment $\overline{x^2}$ (or $\overline{R^4}$), under the following set of conditions which are obtained by eliminating s from Eqs. (28).

$$\left.\begin{aligned}&m = \frac{(\overline{X})^2 - B^2}{\sigma_x^2}\left[= \frac{(\overline{U^n})^2 - A_U}{\overline{U^{2n}} - (\overline{U^n})^2}\right]\ ;\\&m = \frac{\Omega_1^2 - A^2}{\overline{R^4} - \Omega_1^2} \qquad \text{for } X \propto R^2,\\&m = \frac{m_T^2 - E_0^2}{\sigma_T^2} \qquad \text{for } X \propto E\ .\end{aligned}\right\} \quad \cdots\cdots\cdots (29)$$

IV. A fluctuation phenomenon and its number of dimensions

In analyzing a natural phenomenon it is needless to say that one should take it as naturally as possible. Accordingly, when his only concern is a fluctuating state of the phenomenon at hand, the number of dimensions of probability space for the fluctuating state can be chosen independently of the number of dimensions of the original space where the phenomenon really exists. This fact essentially corresponds to the mutual independence between position X of repose and dynamic quantity ΔX, and also to the distinction between "thing in itself" for "thesis" and "phenomenon" for "antithesis". This is the reason why a number N has been used as the number of dimensions in Sections II and III (therefore $m = N/2$). From Eqs. (4) and (5) one can see that the number N is related to the frequency interval W and the time interval T. When the n one-dimensional random variables in the original space where n is the number of dimensions of the original space are mutually independent, the number N obviously coincides with n.

V, The derivation of the Bessel (or gamma) distribution density function from various points of view

(i) Paying attention to the remark in Section IV and noting the independence of a_n's in Eqs. (4), one can derive for an N-dimensional Markoff process X_t the following equations of diffusion (or parabolic) type by using the Smoluchowski equation, the derivation of which is analogous to that of the Fokker-Plank equation.[8]

$$\left.\begin{aligned}&\frac{\partial p}{\partial s} = -a(s)\frac{\partial p}{\partial \xi_1} - d(s)\ \Delta_\xi p \ \ (\text{backward})\\&\frac{\partial p}{\partial t} = -a(t)\frac{\partial p}{\partial x_1} + d(t)\ \Delta_x p \ \ (\text{forward})\end{aligned}\right\} \quad \cdots\cdots\cdots (30)$$

Here $\alpha(t)$ and d(t) are velocities of "mean" and "variance", $\Delta_x \equiv \sum_1^N \partial^2/\partial x^2_i$ etc. P denotes a conditional probability density $P(\vec{x}, t \mid \vec{\xi}, s)$, i. e. a probability that X_t lies in an interval $[\vec{x}, \vec{x}+\vec{dx}]$ ($\vec{x} \equiv [x_1, \cdots\cdots, x_N]$) at time t, given that X_t is equal to $\vec{\xi} \equiv [\xi_1, \cdots\cdots, \xi_N]$ at time s. A solution of Eqs. (30) becomes (cf. Eq. (12) and use Bachelier's method)

$$P(\vec{x}, t \mid \vec{\xi}, s) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi \cdot 2\int_s^t d(u)\, du}} e^{-\frac{\left(x_i - \left[\xi_i + \delta_{i1} \int_s^t a(u)\, du\right]\right)^2}{4 \cdot \int_s^t d(u) du}}$$

$$\equiv \prod_{i=1}^{N} n\left(x_i\, ;\, \xi_i + \delta_{i1}\int_s^t a(u)du,\ 2\int_s^t d(u)du\right). \qquad \cdots\cdots\cdots(31)$$

Hence, comparing the last equation with Eq. (12), one finds

$$A = \int_s^t a(u)du,$$

$$\sigma^2 = 2\int_s^t d(u)du,$$

$$P(R) = \int \cdots\cdots \int_{\omega_N} P(\vec{x}, t \mid \vec{\xi}, s) R^{N-1} d\omega_N \quad (d\omega_N = \text{an incremental N-dimensional solid angle}).$$

For a special case: $a(t) = 0$, $d(t) = d_0$ (constant), using Green's formula in N-dimensional heat conduction, one can directly derive a similar relation as Eq. (31) and finally the first relation of Eqs. (18) with

$$\Omega_1 = 4d_0 m(t-s), \quad m = N/2.$$

(ii) One can also find Eq. (14) by applying the N-dimensional polar coordinate transformations:

$$(x_1, \cdots\cdots, x_N) \longrightarrow (R, \theta_1, \cdots\cdots, \theta_{N-1})$$

to Eq. (12) and using the integral relation :

$$J_\nu(Z) \cdot \Gamma\left(\nu + \frac{1}{2}\right)\sqrt{\pi} = \left(\frac{Z}{2}\right)^\nu \cdot \int_0^\infty e^{iZ\cos\theta} \cdot \sin^{2\nu}\theta d\theta. \qquad \cdots\cdots\cdots(32)$$

(iii) When the N one-dimensional random variables x_k's in Eq. (1), each of which is normally distributed with mean zero and variance σ^2, are mutually independent, one can find from the reproductive property of chi-square distribution[11]

$$V = \sum_{k=1}^{N} (x_k/\sigma)^2 = R^2/\sigma^2$$

follows chi-square distribution with N degrees of freedom. Finally,

$$P(R) = P(V)\frac{dV}{dR}$$

becomes Eq. (18).

(iv) By combining Eqs. (1) and (22) with $\overline{x_k{}^l x_j{}^m}$ (l, m = positive integers) obtained by differentiating a characteristic function

$$m(t_1, \cdots\cdots, t_N) = \exp\left(it_1 A - \frac{\sigma^2}{2}\sum_1^N t_k{}^2\right)$$

of Eq. (12), Eqs. (28) can be derived. Substituting Eqs. (28) in the expansion

$$m(t) \equiv \int_{-\infty}^{\infty} e^{tx} P(X)\, dX \quad (P(X) = 0, X \leqslant 0) = 1 + \bar{x}t + \frac{1}{2!}(\bar{x}^2 + \sigma_x{}^2)t^2 + \cdots\cdots,$$

one obtains the integral equation (27) for P(X) and its solution (23).

(v) In the problem of generalized random flights[7)9)10)] a probability density P(R) of position

$$\vec{R}\equiv[x_1,\cdots\cdots,x_N]=\sum_{j=1}^{s}\vec{r_j}$$

after s independent displacements

$$r_j\equiv[x_{1j},\cdots\cdots,x_{Nj}]$$

is necessary. Using the N-dimensional polar coordinate transformations:

$$P(x_{1j},\cdots\cdots,x_{Nj})\longrightarrow P(r_j,\theta_{1j},\cdots\cdots,\theta_{N-1.j})$$

and a permutation:

$$P(\theta_{1j},\cdots\cdots,\theta_{N-1.j})\prod_{k=1}^{N}d\theta_{kj}\longrightarrow\frac{1}{S_{(N)j}}d\&_{(N)j}$$

derived from the fact that the probability that r_j faces any surface element $d\&_{(N)j}$ of the hypersphere (radius : r_j, surface area : $S_{(N)j}$) is constant, one obtains from Eqs. (11) and (9)

$$F(\lambda) = \left(\Gamma\left(\frac{N}{2}\right)\right)^{s}\cdot\prod_{j=1}^{s}\overline{\left(\frac{2}{\lambda r_j}\right)^{\frac{N}{2}-1}J_{\frac{N}{2}-1}(\lambda r_j)}\quad\left(m=\frac{N}{2}=\nu+1\right), \qquad\cdots\cdots\cdots(33)$$

$$P(R) = \left(\Gamma\left(\frac{N}{2}\right)\right)^{s-1}\cdot\left(\frac{1}{2}\right)^{\frac{N}{2}-1}\cdot\int_0^{\infty}(\lambda R)^{\frac{N}{2}}J_{\frac{N}{2}-1}(\lambda R)$$

$$\cdot\prod_{j=1}^{s}\overline{\left\{\frac{J_{\frac{N}{2}-1}(\lambda r_j)}{\left(\frac{\lambda r_j}{2}\right)^{\frac{N}{2}-1}}\right\}}\,d\lambda\ (\overline{\quad}\ :\ \text{mean with respect to } r_j) \qquad\cdots\cdots\cdots(34)$$

For a special case of s=2, following equations are obtained.

$$\left.\begin{aligned}
P(R) &= \frac{R}{2^{2\nu-1}}\left(\frac{1}{r_1r_2}\right)^{2\nu}\frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{1}{2})}\cdot[R^2-(r_1-r_2)^2]^{\nu-\frac{1}{2}}\cdot[(r_1+r_2)^2-R^2]^{\nu-\frac{1}{2}}\\
P(R) &= \frac{2}{\pi}\ \frac{R}{\sqrt{\{R^2-(r_1-r_2)^2\}\{(r_1+r_2)^2-R^2\}}}\quad(N=2)\\
P(R) &= \frac{R}{2r_1r_2}(N=3)\cdots\cdots\cdots\text{for } |r_1-r_2|<R<|r_1+r_2|.
\end{aligned}\right\}\cdots\cdots\cdots(35)$$

$$P(R)=0 \qquad\cdots\cdots\cdots\text{for } R<|r_1-r_2|,\ |r_1+r_2|<R<\infty. \qquad\cdots\cdots\cdots(36)$$

When

$$r_j=r_{j0}(\text{constant, i.e. } P(r_j)=\delta(r_j-r_{j0})),$$

$$Q(R) = \int_0^R P(R)dR$$

agrees with watson's result and if $R>\sum_{j=1}^{s}r_{j0}$,

$$P(R)=0.$$

Now a problem——$\vec{R}=\sum_{j=0}^{s}\vec{r_j}$ under constraints

$$\vec{r_0}\equiv\vec{A}\ (\text{constant})\ \text{and}\ |\vec{r_j}|\ =\ r(\text{constant}\ ;\ j\neq0)$$

is considered. By the "method of steepest descent[9)]" $F(\lambda)$ given by Eq. (33) is found to agree with Eq. (13) approximately. Accordingly, Eq. (14) can be derived by the procedure stated in Section III.

(vi) Using the method of convolution of distribution and that of steepest descent, the characteristic function

$$m(t_1,\cdots\cdots,t_N)=\prod_{j=1}^{S} m_j(t_1,\cdots\cdots,t_N)e^{it_1A} \text{ of } P(x_1,\cdots\cdots,x_N)\ (\vec{R}\equiv[x_1,\cdots\cdots,x_N])$$

in case (v) can be approximated by that in case (iv), that is, one gets Eq. (12). The rest of procedures to derive P(R) in Eq. (14) is shown in the above.
If P constant vectors $\vec{A}_j$'s are given for $\vec{A}$, Eq. (14) becomes an approximate expression under the relation

$$A=\sqrt{\prod_{j=1}^{P} A_j^2}\quad (|\vec{A}_j|=A_j;\ |\vec{A}|=A).$$

VI. A series solution of the problem of generalized random flights

According to an analogy of the "continuity theorem" by Lévy one will find that when $F(\lambda)$ in Eq. (10) tends to $F_0(\lambda)$ (namely, $F(\lambda)=F_0(\lambda)\,(1+\sum_1 C_n\lambda^n)$) P(R) in Eq. (9) correspondingly tends $P_0(R)$. Now setting $F_0(\lambda)$ equal to $F(\lambda)$ in Eqs. (18), $F(\lambda)$ in Eq. (10) can be expanded as follows.[7]

$$\left.\begin{aligned} F(\lambda)&=e^{-\frac{\Omega_1}{4m}\lambda^2}\cdot\left[1+\sum_{n=2}^{\infty}C_n\lambda^{2n}\right],\quad C_1=0,\\ C_n&=\sum_{k=0}^{n}\frac{(-1)^k\Gamma(m)\Omega_k}{2^{2k}\cdot k!\,\Gamma(m+k)}\cdot\frac{{\Omega_1}^{n-k}}{(n-k)!(4m)^{n-k}} \end{aligned}\right\}\qquad\cdots\cdots\cdots(36)$$

Substituting this formula in Eq. (9), one can derive after some tedious calculations (cf. Eqs. (22))[6]

$$P(R)=\frac{2m^m R^{2m-1}}{\Gamma(m){\Omega_1}^m}e^{-\frac{m}{\Omega_1}R^2}\cdot\left\{1+\sum_{n=2}^{\infty}\sum_{k=0}^{n}\frac{n!(-1)^k\Omega^k m^k}{k!(n-k)!(m)_k{\Omega_1}^k}L_n^{(m-1)}\left(\frac{m}{\Omega_1}R^2\right)\right\},\qquad\cdots\cdots\cdots(37)$$

$$P(X)=\frac{X^{m-1}}{\Gamma(m)S^m}e^{-\frac{X}{S}}\cdot\left\{1+\sum_{n=2}^{\infty}\sum_{k=0}^{n}(-1)^k\,{}_nC_k\frac{\overline{x^k}\cdot m^k}{(m)_k\cdot(\ \ )^k}L_n^{(m-1)}\left(\frac{X}{S}\right)\right\},\qquad\cdots\cdots\cdots(38)$$

$$X=\frac{Sm}{\overline{U^n}}U^n\quad (U=R \text{ for } n=2,\ U=E \text{ for } n=1).$$

Under the conditions: Eqs. (1), (12), the above P(R) (or P(X)) reduces to its own first term as given by Eq. (18) (or Eq. (26)) since

$$\Omega_n=\frac{(m)_n\Omega_1}{m^n}\quad ((m)_n\equiv\Gamma(m+n)/\Gamma(m)).$$

In case of correlated random flights the effect of correlation is taken care of in the calculation :

$$\Omega_k=\overline{R^{2k}}=\left(\sum_1^S\sum_1^S(\vec{r}_j+\vec{A})(\vec{r}_i+\vec{A})\right)^k.$$

For the same problem of S uncorrelated random flights with $A=0$, as in case (v) of

Section V, Eq. (37) becomes ($|\vec{r}_j|=r_0$; $j\neq 0$)

$$\left.\begin{aligned}
P(R) &= \frac{2m^m \; R^{2m-1}}{\Gamma(m)S^m r_0^{2m}} e^{-\frac{m}{sr_0^2}R^2} \cdot \Big[1+\sum_{n=2}^{\infty} 2^{2n}Q_n \\
&\quad \cdot n!\frac{m^n}{s^n r_0^{2n}} L_n^{(m-1)}\Big(\frac{m}{sr_0}R^2\Big)\Big], \\
&\Big(Q_1=0,\; Q_2=-\frac{sr_0^4}{32m^2(m+1)},\quad m=\frac{N}{2}\Big) \\
Q_n &= \sum_{l=0}^{n}\Big[\sum_{\substack{r+s+t+\cdots=s \\ s+2t+3u+\cdots=l}} \frac{s!}{r!s!t!u!\cdots} D_0^r D_1^s D_2^t\cdots\Big] \\
&\quad \cdot \frac{s^{n-l}\cdot r_0^{2n}}{(n-l)!(4m)^{n-l}}, \\
D_n &= \frac{(-1)^n\Gamma(m)}{2^{2n}\cdot n!\Gamma(m+n)};\; \Omega_1=sr_0^2.
\end{aligned}\right\}\quad \cdots\cdots\cdots(39)$$

For a special case when s approaches infinity one can get the following asymptotic expression.[7)9)]

$$\left.\begin{aligned}
P(R) &= \frac{2\left(\frac{N}{2}\right)^{\frac{N}{2}} R^{N-1}}{\Gamma(N/2)\cdot s^{N/2}r_0^N} e^{-\frac{NR^2}{2sr_0^2}} \\
&\quad \cdot \Big\{1-\frac{2}{(N+2)s} L_2^{(m-1)}\Big(\frac{N}{2sr_0^2}R^2\Big)+0\Big(\frac{1}{s^2}\Big)\Big\} \\
L_n^{(\alpha)}(x) &= \frac{e^x x^{-\alpha}}{n!}\left(\frac{d}{dx}\right)^n \{e^{-x}x^{n+\alpha}\} = \sum_{j=0}^{n}\binom{n+\alpha}{n-j}\frac{(-x)^j}{j!}
\end{aligned}\right\}\quad \cdots\cdots\cdots(40)$$

This agrees with the result of Rayleigh when $N=2$ and 3. Similarly, letting $F_0(\lambda)=$ Eq. (13), one can obtain relations corresponding to Eqs. (37), (39) whose first terms are equal to Eq. (14).

VII. Joint probability density function for many kinds of random processes and joint gamma distribution density

So far the discussion has been limited to a single random process, but it is often necessary to consider a joint probability density of the form $P(R_1,R_2,\cdots\cdots,R_k)$ where

$$R_j^2=\sum_{i=1}^{N} x_{ij}^2 (j=1,2,\cdots\cdots,K)$$

for the following three cases :

(i) correlated Brownian motions —— there exist correlations among more than two Brownian processes of particles,

(ii) random phase or amplitude distribution —— as in the case of "diversity reception", one can observe that an electromagnetic wave received simultaneously at more than two distinct points has a random phase or amplitude distribution due to the difference of spatial distance and time, that of propagation constant, and frequency selectivity,

(iii) highly polymerized compounds —— there are correlations among more than two free chains in one or above two highly polymerized compounds. From the integrals (6) and (7) the following expressions are obtained in a similar way as given in Section II (cf. Eqs. (9),(10), and (11))[10)] :

$$P(R_1,R_2\cdots,R_k)=2^{k-\sum^{K}m_j}\cdot\prod_{j=1}^{K}\frac{R_j^{m_j}}{\Gamma(m_j)}\cdot\int_0^{\infty}\cdots\cdots\int_0^{\infty}$$

$$\left[\prod_{j=1}^{K}\lambda_j^{m_j}\cdot J_{m_j-1}(\lambda_jR_j)\right]\quad F(\lambda_1,\lambda_2,\cdots\cdots,\lambda_K)\prod_{j=1}^{K}d\lambda_j, \qquad \cdots\cdots\cdots(41)$$

$$F(\lambda_1,\lambda_2\cdots\cdots,\lambda_K)=\overline{\prod_{j=1}^{K}\Gamma(m_j)\frac{J_{m_j-1}(\lambda_jr_j)}{(\frac{1}{2}\lambda_jr_j)^{m_j-1}}}$$

$$=\frac{1}{\prod_{j=1}^{K}S_{(N_j)}}\int^{\&_{(N_1)}}\cdots\int\int^{\&_{(N_2)}}\cdots\int\cdots\int^{\&_{(N_K)}}\cdots\Big|\left[F(\mu_{11},\cdots,\mu_{1N_1};\cdots;\mu_{K1},\cdots,\mu_{KN_K}\right]_{(\mu_{j1},\cdots\mu_{jN_j})\longrightarrow(\lambda_j,\varphi_{j1},\cdots\cdots\varphi_{jN_j-1})}$$

$$\cdot\prod_{j=1}^{K}d\&_{(N_j)}, \qquad \cdots\cdots\cdots(42)$$

$$F(\mu_{11},\cdots\cdots,\mu_{KN_K})\equiv\overline{\exp i\left[\sum_{j=1}^{N_1}\mu_{1j}x_{1j}+\cdots\cdots+\sum_{p=1}^{N_K}\mu_{Kp}x_{Kp}\right]}$$

$$\lim_{\lambda_j(\forall j)\to 0}F(\lambda_1,\lambda_2,\cdots\cdots\lambda_K)=1,\quad(m_i=N_j/2,j=1,2,\cdots,k), \qquad \cdots\cdots\cdots(43)$$

where

$$S_{(N_j)}=\frac{\sqrt{\pi}^{N_j}\cdot N_j}{\Gamma(1+N_j/2)}$$

and $d\&_{(N_j)}$ denote respectively the surface area and incremental surface area of N_j-dimensional unit hypersphere. When R_j 's are mutually independent, i. e. when

$$P(R_1,\cdots\cdots,R_k)=\prod_{j=1}^{K}P(R_j),$$

Eq. (42) reduces to

$$F(\lambda_1,\cdots\cdots,\lambda_k)=\prod_{j=1}^{K}F(\lambda_j).$$

Specially when K=2 and $N_1=N_2\equiv N$, one can derive after some calculations[2)6)7)10)]

$$P(R_1,R_2)=\frac{4R^m{}_1R^m{}_2\,(\sqrt{\Omega'_{11}\Omega'_{12}\rho_r})^{1-m}}{\Gamma(m)\Omega'_{11}\Omega'_{12}(1-\rho_r)}\cdot e^{-\frac{1}{1-\rho_r}\left[\frac{R_1^2}{\Omega'_{11}}+\frac{R_2^2}{\Omega'_{12}}\right]}\quad I_{m-1}\left(\frac{2\sqrt{\rho_rR_1R_2}}{(1-\rho_r)\sqrt{\Omega'_{11}\Omega'_{12}}}\right)$$

$$=P(R_1)P(R_2)\left\{1+\sum_{n=1}^{\infty}n\,\rho_r^n\,B(n,m)\,L_n^{(m-1)}\left(\frac{R_1^2}{\Omega'_{11}}\right)L_n^{(m-1)}\left(\frac{R_2^2}{\Omega'_{12}}\right)\right\}, \qquad \cdots\cdots\cdots(44)$$

$$F(\lambda_1,\lambda_2)=\frac{\Gamma(m)2^{2(m-1)}(\lambda_1\lambda_2)^{1-m}}{(\sqrt{\Omega'_{11}\Omega'_{12}\rho_r})^{m-1}}\cdot e^{-1/4[\Omega'_{11}\lambda_1^2+\Omega'_{12}\lambda_2^2]}\,I_{m-1}\left(\tfrac{1}{2}\sqrt{\Omega'_{11}\Omega'_{12}\rho_r}\cdot\lambda_1\lambda_2\right),$$

$$[\Omega'_{1i}\equiv\Omega_{1i}/m,\ \Omega_{1i}\equiv\overline{R_i^2},i=1,2], \qquad \cdots\cdots\cdots(45)$$

where each of $P(R_i)$ is given by the first relation of Eqs. (18) when R is replaced with R_i and ρ_r denote the correlation coefficient of R_1^2 and R_2^2. Normalization by

$X_i = S_i\, m\, U_i^{n_i}/\overline{U_i^{n_i}}$ ($U_i=R_i$ for $n_i=2$, $U_i=E_i$ for $n_i=1$, $i=1,2$)

results in a joint gamma distribution corresponding to Eq. (26) as follows.

$$P(x_1,x_2)=\frac{1}{\Gamma(m)s_1s_2(1-\rho_x)}\left(\sqrt{\frac{x_1x_2}{s_1s_2\rho_x}}\right)^{m-1}e^{-\frac{1}{1-\rho_x}\left\{\frac{x_1}{s_1}+\frac{x_2}{s_2}\right\}}I_{m-1}\left(\frac{2\sqrt{\rho_x}}{1-\rho_x}\sqrt{\frac{x_1x_2}{s_1s_2}}\right)$$

$$=\frac{(\sqrt{x_1x_2/-s_1s_2\rho_x})^{m-1}}{\Gamma(m)s_1s_2(1-\rho_x)}e^{-\frac{1}{1-\rho_x}\left\{\frac{x_1}{s_1}+\frac{x_2}{s_2}\right\}}\quad J_{m-1}\left(\frac{2\sqrt{-\rho_x}}{1-\rho_x}\sqrt{\frac{x_1x_2}{s_1s_2}}\right)$$

$$= P(x_1)\ P(x_2)\ \left\{1+\sum_{n=1}^{\infty} n\rho_x^n\ B(m,n)\ L_n^{(m-1)}\left(\frac{x_1}{s_1}\right)\ L_n^{(m-1)}\left(\frac{x_2}{s_2}\right)\right\}$$

$$= P(x_1)\ P(x_2)\ \left\{1+\sum_{n=1}^{\infty}\Gamma(m+n)\Gamma(m)n!\rho_x^n\ T_{m-1}^{(n)}\left(\frac{x_1}{s_1}\right)\ T_{m-1}^{(n)}\left(\frac{x_2}{s_2}\right)\right\}$$

$$= P(x_1)\ P(x_2)\ \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma(m)\rho_x^n}{\Gamma(m+k)}\left(\frac{x_1\,x_2}{s_1\,s_2}\right)^k\ L_{n-k}^{(m+2k-1)}\left(\frac{x_1}{s_2}+\frac{x_2}{s_2}\right),$$

$$\lim_{m\to\infty} P(x_1,x_2) = n(x_1,\ x_2;\ \bar{x}_1,\bar{x}_2;\ \sigma_{x_1},\sigma_{x_2};\ \rho_x)$$

$$=\frac{1}{2\pi\sigma_{x_1}\sigma_{x_2}\sqrt{1-\rho_x^2}}\exp\left\{-\frac{1}{2(1-\rho_x^2)}\left[\left(\frac{x_1-\bar{x}_1}{\sigma_{x_1}}\right)^2+\left(\frac{x_2-\bar{x}_2}{\sigma_{x_2}}\right)^2-2\rho_x\left(\frac{x_1-\bar{x}_1}{\sigma_{x_1}}\right)\left(\frac{x_2-\bar{x}_2}{\sigma_{x_2}}\right)\right]\right\} \quad \cdots\cdots\cdots(46)$$

Here $P(X_i)$ is given by Eq. (26) ($S=S_i$) when X is interchanged with X_i and ρ_X denotes the correlation coefficient of X_1 and X_2. From the same consideration as in case (iv) of Section V Eq. (46) can be obtained as a solution of an integral equation :

$$\iint_0^{\infty} e^{t_1x_1+t_2x_2}P(x_1,x_2)dx_1dx_2=[(1-s_1t_1)(1-s_2t_2)-\rho_xt_1t_2s_1s_2]^{-m}. \quad \cdots\cdots\cdots(47)$$

Eqs. (44), (46) have the following properties.

(i) When $m=P/2$ ($P=1, 2, \cdots\cdots$) and $S_i=2$, Eqs. (46) and (47) express respectively the joint chi-square probability density ($P\chi^2$ (X_1, X_2)) with P degrees of freedom and its moment generating function.

(ii) When P pairs of (x_{1j}, x_{2j}) ($j=1,2,\cdots\cdots,P$) follow n ($x_{1j}, x_{2j}; a_j, b_j; \sigma_{x1j}, \sigma_{x2j}; \rho_0$) ($X_1$, X_2) where

$$X_1 = \sum^{p}(x_{1j}-a_j/\sigma_{x1j})^2,$$

$$X_2 = \sum^{p}(x_{2j}-b_j/\sigma_{x2j})^2$$

have the probability density $P\chi^2(X_1, X_2)$ with P degrees of freedom in case (i) ($\rho_X=\rho_0^2$, a constant).

(iii) When $a_j=b_j=0$ and σ_{Xij} are constant in case (ii), (R_1, R_2) where

$$R_i^2=\sum_{j=1}^{p}x_{ij}^2\ (i=1,2)$$

have the joint probability density function expressed by Eq. (44) with

$m=P/2$,

$\rho_r=\rho_0^2$,

$\Omega_{1i}=\overline{R_i^2}=P\cdot\sigma^2x_{ij}$

Allowing approximation by the first and second moment, one may use Eq. (44) for any nonnormal random variables under the constraint

$$m=\Omega_{1i}^2\Big/\overline{(R_i^2-\Omega_{1i})^2}\quad(i=1,2).$$

(iv) When n pairs of (X_{1K}, X_{2K}) ($K=1,\cdots\cdots,n$) have respectively joint probability density functions $P(X_{1K}, X_{2K})$ of the same form as Eq. (46) under the conditions

$m=m_k$,

$S_i=$constant ($i=1,2$),

$\rho_x=$constant,

(X_1, X_2) where

$$X_i = \sum_{k=1}^{n} X_{ik} \ (i=1,\ 2)$$

have the joint probability density given by Eq. (46) with

$$m = \sum_{k=1}^{n} m_k.$$

This shows the reproductive property of joint gamma distribution and similarly $P\chi^2(X_1 X_2)$ in case (i) has the same reproductive property[11] as chi-square distribution shows.

(v) When there is a constant correlation given by

$\Psi(\tau)=\rho_0 \cdot \sigma^2$ (power spectral density : $h(\omega)$),

$\sigma x_{ij} \equiv \sigma^2$,

between x_{1j} and x_{2j} in case (iii), under the ergodic assumption and with the help of the Wiener-Khinchine theorem, autocorrelation function $\phi(\tau)$ and power spectral density $H(\omega)$ of $R(t)$ (continuous stationary time series, $R_1=R(t)$, $R_2=R(t+\tau)$) are expressed as (see Eqs. (44) and (45))

$$\phi(\tau)=\frac{2\pi\sigma^2}{\{B(m,\frac{1}{2})\}^2}\,{}_2F_1\left(-\frac{1}{2},\ -\frac{1}{2};\ m;\ \frac{\Psi(\tau)^2}{\sigma^4}\right) \quad m=P/2, p=1,2,\cdots, \qquad \cdots\cdots\cdots(48)$$

$$H(\omega) = \frac{\sigma^2}{\{B(m,\frac{1}{2})\}^2} \cdot \int_{-\infty}^{\infty} {}_2F_1\left(-\frac{1}{2},\ -\frac{1}{2};\ m;\ \frac{1}{\sigma^4}\int_{-\infty}^{\infty} h(\omega)*h(\omega)e^{i\omega\tau}d\omega\right) e^{-i\omega\tau}d\tau \qquad \cdots\cdots\cdots(49)$$

$$=\underbrace{\frac{\sigma^2}{\{B(m,\frac{1}{2})\}^2}\delta(\omega)}_{\text{(D. C. Component)}}+\underbrace{\frac{\pi/2}{m\sigma^2\{B(m,\frac{1}{2})\}^2}h(\omega)*h(\omega)+\cdots\cdots\cdots}_{(\longrightarrow\text{A. C. Component}\longrightarrow)}. \qquad \cdots\cdots\cdots(50)$$

Specially when P=2, Eq. (48) coincides with the result of Lawson and Uhlenbeck.

VIII. Applications to stochastic processes in physics

The above expressions for $P(R)$ and $P(R_1, R_2)$ are useful to analyze noise, fading or highly polymerized compound, when R or R_1, R_2 denotes envelope amplitude or size of the highly polymerized compound (N=any positive integer, see Section III).

Furthermore, Eqs. (18) explain the Brownian motion of colloidal particle when R is an arrival distance after t seconds (Ω_1=2NDt, D=diffusion coefficient), while the velocity distribution of Maxwell-Boltzmann when R is the momentum in three-dimensional velocity space. The poisson distribution density function or L. Smith's expression in quantum noise may be regarded as a discrete version of Eq. (18) or Eq. (14).

The parameters m, Ω_1, m_T, etc. in the above expressions of probability density can be estimated by the theory of estimation[11]. For instance, the estimates of parameters m and Ω_1 for P(R) in Eqs. (18) can be obtained by the "method of maximum likelihood" corresponding to the "method of moment" for Eq. (29) (or case (iii) of Section VII). One can easily show the maximum likelihood estimates for Ω_1 and m are

$$\Omega_1=\frac{1}{n}\sum_{i=1}^{n} R_i^2 \ \text{(sample mean of } R_i^2\text{)}, \qquad \cdots\cdots\cdots(51)$$

$$\log m-\Psi(m) = \log\left(\Omega_1\Big/\sqrt[n]{\prod_{i=1}^{n} R_i^2}\right) \qquad \cdots\cdots\cdots(52)$$

$$= \log \frac{(\text{arithmetic mean of } R_i^2)}{(\text{geometric mean of } R_i^2)}, \qquad \cdots\cdots\cdots(53)$$

where $(R_1, R_2, \cdots\cdots, R_n)$ are random samples and $\Psi(m)$ denotes Di-gamma function.

R. A. Fisher has shown that maximum likelihood estimate contains every information on unknown parameters which samples can produce.

On the other hand, by using "Lagrange's method of indeterminate coefficients" and "calculus of variation" one can show that the gamma distribution density P(E) given dy Eq. (21) corresponding to the above-mentioned P(R) (in Eqs. (18)) has a maximum entropy under the conditions:

(i) when $E \leqslant 0$, $P(E) = 0$,

(ii) means of energy $\overline{E}$ $(=m_T)$, and its db value $\overline{10\log_{10}(E/m_T)}$, remain constant.

The gamma distribution density (of. Eq. (21) or (26)) is essentially given by the first term of Laguerre expansion for nonnegative functional, while normal distribution density is given as the first term of corresponding Hermite expansion, therefore Eq. (38) (or Eq. (37)) may be considered to correspond to Gram-Charler's polynomial for random variables defined over $(-\infty, \infty)$.

Acknowledgment

The authors would like to thank Prof. I. Takahashi of kyoto University for helpful discussions.

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A most part of this paper was published in the Proc. of the 6th Japan National Congress for Applied Mechanics[12] (1956), pp 511 — 516.

（昭和43年4月16日受理）